

BERNOULLI CONVOLUTIONS AND DIFFERENTIABLE FUNCTIONS

BY

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ABSTRACT. Bernoulli convolutions, similar in structure to convolutions with a constant ratio, are considered in relation to differentiable transformations. A space of functions on the Cantor set leads to highly singular measures that nevertheless resemble absolutely continuous measures sufficiently to control their Fourier-Stieltjes transforms.

To each number θ in $(0, 1)$ there corresponds a measure μ whose Fourier-Stieltjes transform $\hat{\mu}$ is defined by $\hat{\mu}(u) = \prod_1^\infty \cos(u\theta^n)$. Moreover, $\hat{\mu}$ vanishes at infinity unless θ^{-1} is a PV number and $\theta \neq \frac{1}{2}$ [6, pp. 36–41], [8, pp. 147–152]. The measure μ has an interesting property apparently not observed until now, and this property is shared by measures close to μ in structure. Operating in a certain space of functions on a Cantor set, we can produce highly singular measures with the same property.

Let $S = (\theta_n)_1^\infty$ be a sequence of numbers in $(0, 1)$, and let $\theta = \lim \theta_n$, $0 < \theta < 1$, fulfill the condition for the vanishing of $\hat{\mu}^{(\infty)}$ mentioned before. We define λ by the formula $\hat{\lambda}(u) = \prod_1^\infty \cos(u\theta_1 \cdots \theta_n)$, observing that λ is carried by the set $\Sigma(S)$ of all infinite sums $\Sigma \pm \theta_1 \cdots \theta_n$.

THEOREM 1. *The measure λ has the property*

$$(R_1) \quad \lim \int \exp iu\varphi(t) \cdot \lambda(dt) = 0$$

for every φ in $C^1(-\infty, \infty)$ with $\varphi' > 0$ everywhere.

THEOREM 2. *For each Hausdorff measure-function h , there is a probability measure λ_1 with property (R_1) , whose support has h -measure 0 and is contained in $\Sigma(S)$.*

1. The next lemma contains a substantial part of the analysis in the theorems. To each $A > 1$ we denote by $\hat{\lambda}_A(u)$ the partial product in $\hat{\lambda}(u)$, extended over indices n for which $|\theta_1 \cdots \theta_n u| < A$.

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LEMMA. To each $\epsilon > 0$ there is an A so large that $|\hat{\lambda}_A(u)| < \epsilon$ when $|u| > A$.

To prove this we choose A so that $|\hat{\mu}(u)| < \epsilon$ when $|u| > A$. For large u $\hat{\lambda}_A(u) = \prod_1^\infty \cos(u\varphi_1 \cdots \varphi_n)$, wherein $A \leq v < A\theta^{-2}$ and the numbers $\varphi_1, \dots, \varphi_n, \dots$ belong to the tail of the sequence S —in fact they belong to $(\theta_n)_p^\infty$, with $\theta_1 \cdots \theta_p |u| < A$. The infinite product is then uniformly close to $\hat{\mu}(v)$, for large u , and this proves the lemma.

Instead of giving the proof of Theorem 1 immediately, we introduce some machinery leading to Theorem 3, a stronger result. Let C be the Cantor set of sequences $x = (x_\kappa)_1^\infty$, $x_\kappa = -1, +1$. C is a topological group and the Haar measure σ is the customary product measure. The distance $d(x, x') = 2^{-N}$ if $x = x'_1, \dots, x'_N = x'_N$ but $x_{N+1} \neq x'_{N+1}$. The modulus of continuity $\omega_\kappa(f)$ of a mapping f of C into a metric space—in particular into C itself—is given by

$$\omega_\kappa(f, x) = \sup d(f(x'), f(x)): d(x, x') \leq 2^{-\kappa}.$$

The space λ_1^0 is composed of continuous mappings of C into itself such that $2^\kappa \omega_\kappa(f) \rightarrow 0$ in σ -measure, while λ_1^p ($1 \leq p < \infty$) is defined by the same relation in $L^p(\sigma)$. These spaces are complete metric groups.

C is mapped onto $\Sigma(S)$ by the transformation $Y(x) = \Sigma \theta_1 \cdots \theta_\kappa x_\kappa$ and this also maps the measure σ onto λ , i.e. $\lambda = \sigma \circ Y^{-1}$ as set-functions.

THEOREM 3. Let I denote the identity map of C , let f belong to λ_1^0 , and let $I + f$ denote a sum in the group C . Then $Y_1 = Y_0(I + f)$ transforms σ onto a measure λ_1 with property (R_1) .

PROOF. For large u we define $n(u) = n$ to be the least integer such that $|\theta_1 \cdots \theta_n u| < A$ (A is fixed until the end of proof). Then we divide C into cylinders C_j ($1 \leq j \leq 2^n$) of σ -measure 2^{-n} . Of course these are just cosets of the subgroup $x_1 = \cdots = x_n = +1$. Now $\omega_n(f)$ is constant on each C_j —let its value be 2^{-nj} . On C_j we can write

$$Y(x + f(x)) = Y(x + \bar{x}_j) + O(\theta_1 \cdots \theta_{n_j}),$$

for some element $\bar{x}_j$ of C . Now $n_j - n \rightarrow +\infty$ except on sets C_j whose total σ -measure tends to 0, and when $n_j - n$ is large, then $|u| \theta_1 \cdots \theta_{n_j}$ is correspondingly small. Therefore we can focus on the estimation of $\exp iu\varphi \circ Y(x + \bar{x}_j)$, and observe that uniform continuity of φ' gives

$$u\varphi \circ Y(x + \bar{x}_j) = u\varphi'(\bar{y}_j) \cdot Y(x + \bar{x}_j) + \bar{z}_j + o(1),$$

where $\bar{y}_j$ belongs to $\Sigma(S)$, whence $\varphi'(\bar{y}_j) \geq \eta > 0$. Thus the integral over C_j has absolute value $< o(2^{-n}) + 2^{-n} |\hat{\lambda}_A(v)|$, with $|v| \geq \eta |u|$. By the lemma we can make this $< \epsilon 2^{-n}$ by increasing A and $|u|$, and this proves the theorem.

THEOREM 2'. *For each Hausdorff measure-function h , $I + f$ maps C onto a set of h -measure 0 in C , except for a subset of first category of λ_1^p .*

In the proof we use operators T_κ , defined by

$$T_\kappa f(x) = f(x_1, \dots, x_\kappa, 1, 1, \dots).$$

Clearly $\omega_N(T_\kappa f - f) \leq \omega_N(f)$, and of course

$$\omega_N(T_\kappa f - f) \leq \sup d(T_\kappa f(x), f(x)).$$

Therefore $T_\kappa f \rightarrow f$ uniformly as $\kappa \rightarrow +\infty$ and moreover

$$\sup_N 2^N \|\omega_N(T_\kappa f - f)\|_p \rightarrow 0.$$

Thus λ_1^p contains a dense subgroup of functions f with a finite range. Suppose, for definiteness, that f is constant on sets of diameter 2^{-r} and let $N > r$, $N = N_1 < M_1 < \dots < N_q < M_q < \dots$, $1 \leq q \leq 2^N$. We divide C into the standard sets C_q of measure 2^{-N} and define

$$\psi(x) = T_N(x) + T_M(x) \quad \text{on } C_q (N = N_q, M = M_q).$$

In estimating $\|\omega_\kappa(\psi)\|_p$ we first consider $\kappa = 1, 2, \dots, N_1$. Now $d(\psi, 1) \leq 2^{-N}$ everywhere and $d(\psi, 1) \leq 2^{-N}2$ except on C_1 . But C_1 meets exactly one of the cylinders of index κ , so $\|\omega_\kappa \psi\|_p \leq 2^{-N}2^{-\kappa/p} + 2^{-N}2$. This is small in comparison with $2^{-\kappa}$, provided N and $N_2 - N_1$ are large. When $N_1 \leq \kappa \leq M_1$ we find $\|\omega_\kappa(\psi)\|_p \leq 2^{-\kappa}2^{-N/p} + 2^{-N}2$, and this can be made small enough by increasing N and $N_2 - M_1$. When $\kappa \geq M_1$, the argument remains the same, except that $\omega_\kappa(\psi) = 0$ on C_1 , etc. Thus ψ has a small norm in λ_1^p , for appropriate choices of $N, N_1, M_1, \dots$.

It remains to investigate the mapping properties of $f(x) + x + \psi(x)$. On C_q , this equals $f(x) + x + T_M(x) + T_N(x)$. Using the fact that f is constant on sets of diameter 2^{-r} , and $N = N_q > r$, we see that C_q is mapped into at most 2^N sets of diameter 2^{-M} . By making the sum $\sum 2^N q \cdot h(2^{-M}q) < \eta$ (say), we obtain a function $f + \psi$, close to f in λ_1^p , such that $I + f + \psi$ transforms C into a union of sets B_s , with $\sum h(\text{diam } B_s) < \eta$. Moreover, this remains true for all f^* in a neighborhood W^* of $f + \psi$, because the metric in λ_1^p is stronger than the uniform metric. Taking a sequence $\eta_r \rightarrow 0$, we obtain a dense G_δ -set in λ_1^p with the property that $(I + f)C$ has h -measure 0.

Theorem 2' leads directly to Theorem 2 because Y is uniformly continuous: to each h there is an h_1 so that $h\text{-mes } Y(F) \leq h_1\text{-mes } F$ for every subset F of C .

2. In the definition of property (R_1) , the continuity of φ' is essential. If, for example, we allow all absolutely continuous functions φ , with $1 < \varphi' < 2$ a.e., then the corresponding measures would necessarily be absolutely continuous [6]. Therefore it is interesting to state a theorem in which φ' need not be continuous,

nor even locally integrable. For this kind of theorem we take $\theta_n \equiv 2$; although this can be avoided, the complications are great. Compare [1], [7].

THEOREM 4. *Suppose that $\varphi(t)$ is defined on $[-1, 1]$ and is differentiable almost everywhere with derivative $\varphi' \neq 0$. Let $f \in \lambda_1^0$. Then for almost all y in C , the mapping $H_y: H_y(x) = \varphi \circ Y(x + y + f(x))$ transforms the measure σ onto a measure λ_1 such that $\hat{\lambda}_1(\infty) = 0$.*

The proof depends on Marcinkiewicz' variant of Lusin's theorem [8, pp. 73–77]: to each $\epsilon > 0$ there is a function φ_* of class $C^1[-1, 1]$, such that $m\{\varphi \neq \varphi_*\} < \epsilon$. Now $\varphi' = \varphi'_*$ a.e. on $\{\varphi = \varphi_*\}$ so there is a closed set, of measure $> 2 - \epsilon$, on which $\varphi'_* \neq 0$ and $\varphi = \varphi_*$. By parabolic interpolation [8] we can adjust φ_* off the closed set so that it remains in $C^1[-1, 1]$ and $\varphi'_* \neq 0$ except on a countable set.

We claim that

$$\lim \int \exp iu\varphi_* \circ Y(x + y + f(x))\sigma(dx) = 0$$

for every y in C . Indeed, if $J \subseteq (-1, 1)$ is an interval on which $\varphi_* > 0$ or $\varphi_* < 0$, then φ_* can be extended from J to a function Φ of class $C^1(-\infty, \infty)$, with $\Phi' > 0$ or $\Phi' < 0$ everywhere. By Theorem 2 and a known theorem [8, p. 145], $\lim \int_J (\cdot) = 0$. Since $\varphi'_* = 0$ only on a countable closed set, this is sufficient to establish our claim.

Now the σ -measure of the x -set

$$\varphi \circ Y(x + y + f(x)) \neq \varphi_* \circ Y(x + y + f(x))$$

is a Lebesgue-measurable function of y , whose integral over C is $< \epsilon$. So the σ -measure is $< \epsilon^{1/2}$, except for a y -set of measure $< \epsilon^{1/2}$. Under this bound on the σ -measure

$$\limsup \left| \int \exp iu\varphi \circ Y(x + y + f(x))\sigma(dx) \right| < \epsilon^{1/2}.$$

Since ϵ was arbitrary, the limit is 0 for almost all y . In these arguments we used the observation that λ , the distribution of the function Y , is $dt/2$ on $(-1, 1)$, and combined this with Fubini's theorem.

Differentiability of φ' is not the weakest condition for Marcinkiewicz' interpolation process [8, p. 228] and the theorems cited lead to a stronger version of Theorem 4. If we allow λ to be singular, for example by taking $\theta < 1/2$, then φ' must exist a.e. for λ . Marcinkiewicz' theorem becomes quite technical in this situation.

3. In this final section we suppose $\theta_n = \theta < 1/2$ for all n and write $\Sigma(\theta)$ for the support of the measure μ . If M is an infinite set of positive integers, then

$\Sigma(\theta, M)$ stands for the set of sums $\Sigma x_n \theta^n$ in which $x_n = -1, +1$ and $x_n = +1$ for $n \notin M$. Then $\Sigma(\theta, M)$ carries a measure λ whose Fourier-Stieltjes transform is

$$\prod_M \cos u \theta^n \times \prod_{M'} \exp -iu \theta^n.$$

As we observed earlier $\Sigma(\theta)$ admits decompositions into 2^n cylinders; their diameter is exactly $2\theta^{n+1}(1-\theta)^{-1}$ and their distances at least $[1-\theta(1-\theta)^{-1}] \cdot 2\theta^n$. When M' contains segments of unbounded length, then some function φ of class $C^1(-\infty, \infty)$, with $\varphi' > 0$, transforms $\Sigma(\theta, M)$ onto a set of uniqueness [3, 2VII]. Incidentally, this is also true when $\theta = \frac{1}{2}$, but then $\Sigma(\frac{1}{2}, M)$ is a set of uniqueness as soon as M' is infinite. Let us say that M is *deficient* if M' contains segments of unbounded length.

THEOREM 5. *Let h be a positive function on $(0, 4)$ and $h(0+) = 0$. Then for a certain deficient sequence M the product measure λ on $\Sigma(\theta, M)$ has this property:*

$$\lim \int \exp -iu\varphi(t) \cdot \lambda(dt) = 0,$$

whenever $\varphi \in C^1[-2, 2]$, $\varphi' > 0$ everywhere, and $|\varphi'(a) - \varphi'(b)| \leq h(|b-a|)$ for all a, b in $[-2, 2]$.

Sets with the two properties claimed for $\Sigma(\theta, M)$ were first constructed in [4], by a complex process; the examples given below are essentially variants of Theorem 1.

By the hypothesis on h , there is a function $\Gamma(u) > 0$, defined and increasing without limit for $u > 1$, such that $\Gamma(u)h(u^{-1}\Gamma(u)) \rightarrow 0$. It is convenient to set $\Gamma(u) = \Gamma(|u|)$ if $u < -1$.

Now we are led to define the infinite product $\lambda_*(u) = \prod \cos u \theta^n$: $\theta^n |u| < \Gamma(u)$. We shall prove that if $\lim \lambda_* = 0$, then λ has the property claimed in Theorem 5. First we choose $\delta > 0$ so that $\delta < \varphi' < \delta^{-1}$ on $[-2, 2]$, and define $n = n(u)$ to be the largest integer such that $|u|\theta^n \geq \delta^{-1}\Gamma(\delta^{-1}u)$. Now $\Sigma(\theta)$ is covered by cylinders of order n , of length $< \theta^{n+1}(1-\theta)^{-1} \leq 2\delta u^{-1}\Gamma(\delta^{-1}u) = L$, say. Here $uLh(L) \rightarrow 0$ by the characteristic property of Γ . Thus φ is represented on each cylinder by a linear function $at + b$, with an error $o(u^{-1})$, and $\delta < a < \delta^{-1}$. The integral of $\exp -iuat$ over a cylinder has magnitude $2^{-n} \prod_n |\cos u \theta^m|$. The latter product is no larger than $|\lambda_*(au)|$, because the inequality $\theta^m |au| < \Gamma(au)$ leads to $\theta^m |u| < \delta^{-1}\Gamma(\delta^{-1}u)$, so $m \geq n$. Since λ_* vanishes at infinity, $\lim \int \exp -iu\varphi \cdot d\sigma = 0$.

To finish the proof of Theorem 5, we exhibit a deficient set M such that λ_* vanishes at infinity. The complement M' will be a union of segments $p_r \leq n \leq p_r + r$ with $p_1 > 1$ and $p_{r+1} > 3r + p_r$ ($r = 1, 2, 3, \dots$). The characteristic

property of θ , that $\theta^{-1} \notin PV$, is equivalent to $\Pi_1^\infty \cos u\theta^{-n} = 0$ for every $u \neq 0$ (see the references in the introduction). By Dini's theorem on monotone sequences, the convergence to 0 is uniform on every compact subset of the positive line. Thus there are numbers $Q = Q(r)$ so that $\Pi_1^Q |\cos u\theta^{-n}| < r^{-1}$ for $\theta \leq u \leq \theta^{-r}$. Now let $p_1, p_2, \dots, p_r$ be defined so that for $r = 1, 2, 3, \dots$

$$p_r > p_{r-1} + r + Q(2r) \quad \text{and} \quad \Gamma(\theta^{-p_r}) > \theta^{-Q(2r)}\theta^{-2r}.$$

In proving that λ_* vanishes at infinity we define $s = s(u)$ by the inequality $\theta < \theta^s u \leq 1$ ($u > 1$) and examine two possible inequalities for s in relation to M' : either $p_r + 2r \leq s < p_{r+1}$ for a certain r , or $p_r < s < p_r + 2r$. One of these must obtain, as soon as $s > p_1$. In the first case $\lambda_*(u)$ contains all factors $\cos u\theta^n$ in which $p + r < n < s$ and $u\theta^n \leq \Gamma(u)$. In particular, this extends to all n such that $1 \leq s - n \leq r - 1$ and $\theta^{n-s} \leq \Gamma(u)$. In different terms $\lambda_*(u)$ contains all factors $\cos u\theta^s \cdot \theta^{-q}$ wherein $1 \leq q < r - 1$, $\theta^{-q} \leq \Gamma(u)$. Now $\theta < u\theta^s \leq 1$, while r and $\Gamma(u)$ increase without bound, so this situation is covered by Dini's theorem.

In the case $p_r < s < p_r + 2r$, $\lambda_*(u)$ contains all factors wherein $p_{r-1} + r \leq n < p_r$ and $u\theta^n < \Gamma(u)$. Now $u\theta^n \leq \theta^{n-s}$, and $\Gamma(u) \geq \Gamma(\theta^{-1}\theta^{-s}) > \theta^{-Q(2r)}\theta^{-2r}$. Hence the inequalities on n are fulfilled if $p_r - Q(2r) \leq n < p_r$. Writing $u\theta^n = \theta^{n-p_r}u\theta^{p_r}$, we observe that

$$|\lambda_*(u)| \leq \prod_1^{Q(2r)} |\cos v\theta^n|, \quad \text{with } \theta \leq v \leq \theta^{-2r},$$

so $|\lambda_*(u)| < r^{-1}$. This completes the proof that $\lambda_*(\infty) = 0$, and also the proof of Theorem 5.

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